A class of bistochastic positive optimal maps in $M_d(\mathbb{C})$

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Abstract

We provide a straightforward generalization of a positive map in $M_3(\mathbb{C})$ considered recently by Miller and Olkiewicz [5]. It is proved that these maps are optimal and indecomposable. As a byproduct we provide a class of PPT entangled states in $d \otimes d$.

Positive maps in matrix algebras play important role both in mathematics and theoretical physics [1, 2, 3, 4]. In the recent paper [5] paper Miller and Olkiewicz considered a linear map $\Lambda_3 : M_3(\mathbb{C}) \to M_3(\mathbb{C})$ ($M_d(\mathbb{C})$ denotes a matrix algebra of $d \times d$ complex matrices) defined as follows

$$\Lambda_3 \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(a_{11}+a_{22}) & 0 & \frac{1}{\sqrt{2}}a_{13} \\ 0 & \frac{1}{2}(a_{11}+a_{22}) & \frac{1}{\sqrt{2}}a_{32} \\ \frac{1}{\sqrt{2}}a_{31} & \frac{1}{\sqrt{2}}a_{23} & a_{33} \end{pmatrix} \geq 0 \ . \tag{1}$$

It was proved [5] that Λ_3 is a bistochastic positive extremal (even exposed) non-decomposable map. In this paper we provide the following generalization $\Lambda_d : M_d(\mathbb{C}) \to M_d(\mathbb{C})$:

$$\Lambda_d(A) = \frac{1}{d-1} \begin{pmatrix} \sum_{i=1}^{d-1} a_{ii} & \cdots & 0 & 0 & \sqrt{d-1}\,a_{1d} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \cdots & \sum_{i=1}^{d-1} a_{ii} & 0 & \sqrt{d-1}\,a_{d-2,d} \\ 0 & \cdots & 0 & \sum_{i=1}^{d-1} a_{ii} & \sqrt{d-1}\,a_{d,d-1} \\ \sqrt{d-1}\,a_{d1} & \cdots & \sqrt{d-1}\,a_{d,d-2} & \sqrt{d-1}\,a_{d-1,d} & (d-1)\,a_{dd} \end{pmatrix}, \tag{2}$$

where $A = [a_{ij}] \in M_d(\mathbb{C})$.

Proposition 1. *Λ_d is a positive map.*

Proof: let $y = \begin{pmatrix} \mathbf{x} \\ x_d \end{pmatrix} \in \mathbb{C}^d, \mathbf{x} \in \mathbb{C}^{d-1}$ and $P_i = |i\rangle\langle i|$ for $i = 1, \ldots, d-1$. One has

$$\Lambda_d\left(yy^\dagger\right) = \frac{1}{d-1}\left(\begin{array}{c|c} \|x\|^2\,\mathbb{I}_{d-1} & \sqrt{d-1}\left(x_d P_1 \mathbf{x} + \sum_{i=2}^{d-1} \bar{x}_d P_i \bar{\mathbf{x}}\right) \\ \hline \sqrt{d-1}\left(x_d P_1 \mathbf{x} + \sum_{i=2}^{d-1} \bar{x}_d P_i \bar{\mathbf{x}}\right)^\dagger & (d-1)\,|x_d|^2 \end{array}\right).$$

Now we use the well known result [2]: a block matrix

$$\left(\begin{array}{c|c} A & B \\ \hline B^\dagger & C \end{array}\right),$$

with $C > 0$ is positive iff

$$A \geq BC^{-1}B^\dagger\ . \tag{3}$$

Hence, to prove that $\Lambda_d\left(yy^\dagger\right) \geq 0$ it is necessary and sufficient to show that

$$|x_d|^2\,\|x\|^2\,\mathbb{I}_{d-1} - \left(x_d P_1 \mathbf{x} + \sum_{i=2}^{d-1} \bar{x}_d P_i \bar{\mathbf{x}}\right)\left(x_d P_1 \mathbf{x} + \sum_{i=2}^{d-1} \bar{x}_d P_i \bar{\mathbf{x}}\right)^\dagger \geq 0.$$

One has

$$\begin{aligned}
&\left(x_d P_1 \mathbf{x} + \sum_{i=1}^{d-1} \bar{x}_d P_i \bar{\mathbf{x}}\right)\left(x_d P_1 \mathbf{x} + \sum_{i=1}^{d-1} \bar{x}_d P_i \bar{\mathbf{x}}\right)^\dagger \leq \left\|\left(x_d p_1 x + \sum_{i=2}^{d-1} \bar{x}_d p_i \bar{x}\right)\right\|^2 \mathbb{I}_{d-1} \\
&= |x_d|^2\left(\|P_1\mathbf{x}\|^2 + \sum_{i=2}^{d-1}\|P_i\bar{\mathbf{x}}\|^2\right)\mathbb{I}_{d-1} = |x_d|^2\,\|\mathbf{x}\|^2\,\mathbb{I}_{d-1},
\end{aligned}$$

which ends the proof. $\square$

Remark 1. *It is very easy to check that Λ_d is unital and trace-preserving and hence it defines a positive bistochastic map.*

Proposition 2. *Λ_d is nondecomposable.*

Proof: to prove it we construct a PPT state ρ_{PPT} such that $\mathrm{Tr}(W_d\rho) < 0$, where $W_d = (\mathbb{1} \otimes \Lambda_d)P_d^+$ denotes the corresponding entanglement witness ([6] and the recent review [4]). Let us define

$$\rho = \left(\begin{array}{c|c|c|c|c}
\sqrt{d-2}e_{11} + e_{dd} & 0 & \cdots & 0 & -e_{1d} \\ \hline
0 & \sqrt{d-2}e_{22} + e_{dd} & \cdots & 0 & -e_{2d} \\ \hline
\vdots & \vdots & \ddots & \vdots & \vdots \\ \hline
0 & 0 & \cdots & \sqrt{d-2}e_{d-1,d-1} + e_{dd} & -e_{d-1,d}^T \\ \hline
-e_{d1} & -e_{d2} & \cdots & -e_{d,d-1}^T & \mathbb{I} - \left(1 - \sqrt{d-2}\right)e_{dd}
\end{array}\right),$$

where $e_{ij} = |i\rangle\langle j|$. Let us observe that $\rho \geq 0$ iff the following $d-1 \times d-1$ submatrix

$$\left(\begin{array}{ccccc} \sqrt{d-2} & 0 & \cdots & 0 & -1 \\ 0 & \sqrt{d-2} & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sqrt{d-2} & -1 \\ -1 & -1 & \cdots & -1 & \sqrt{d-2} \end{array} \right) \geq 0 \ , \tag{4}$$

which is the case due to the fact that its eigenvalues read: $\{\lambda_1 = 0, \lambda_2 = \sqrt{d-2}, \lambda_3 = 2\sqrt{d-2}\}$, where λ_1, λ_3 are simple and λ_2 has multiplicity $d-3$. Consider now the partial transposed

$$\rho^\Gamma = \left(\begin{array}{c|c|c|c|c} \sqrt{d-2}e_{11} + e_{dd} & 0 & \cdots & 0 & -e_{d1} \\ \hline 0 & \sqrt{d-2}e_{22} + e_{dd} & \cdots & 0 & -e_{d2} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & \sqrt{d-2}e_{d-1,d-1} + e_{dd} & -e^T_{d,d-1} \\ \hline -e_{1d} & -e_{2d} & \cdots & -e^T_{d-1,d} & \mathbb{I} - \left(1 - \sqrt{d-2}\right) e_{dd} \end{array} \right) .$$

Its positivity follows from the simple observation that the following 2×2 submatrices

$$\left(\begin{array}{cc} \sqrt{d-2} & -1 \\ -1 & \sqrt{d-2} \end{array} \right) \quad \text{and} \quad \left(\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) \tag{5}$$

are positive. Now,

$$\mathrm{Tr}\,(W_d \rho) = 2\,(d-1) \left(\sqrt{d-2} - \sqrt{d-1} \right) < 0 \ ,$$

which finally proves that Λ_d is nondecomposable. □

Now we are ready to show that a map Λ_d is optimal [7].

Proposition 3. *Λ_d is optimal.*

Proof: to prove optimality we use the following result from [7]: if the entanglement witness $W = (\mathbb{1} \otimes \Lambda) P^+_d$ allows for a set of product vectors $\psi_k \otimes \phi_k$ such that

$$\langle \psi_k \otimes \phi_k | W | \psi_k \otimes \phi_k \rangle = 0 \ , \tag{6}$$

then if $\psi_k \otimes \phi_k$ span $\mathbb{C}^d \otimes \mathbb{C}^d$ the map Λ is optimal. Now, take arbitrary $x \in \mathbb{C}^d$ and define

$$W_d(x) = \mathrm{Tr}_1(W_d \cdot |x\rangle\langle x| \otimes \mathbb{I}_d). \tag{7}$$

One finds

$$W_d(x) = \left[\begin{array}{c|c} z I_{d-1} & \vec{a} \\ \hline \vec{a}^\dagger & u \end{array} \right], \tag{8}$$

where

$$a_i = \sqrt{d-1} \cdot \left\{ \begin{array}{lll} x_d^* x_i & \text{for} & i < d-1 \\ x_d x_i^* & \text{for} & i = d-1 \end{array} \right. \ ,$$

$z = \sum_{i=1}^{d-1} |x_i|^2$ and $u = (d-1)|x_d|^2$. Note that $W_d(x)$ is at least of rank $d-1$ and hence its kernel is at most 1-dimensional. To find the corresponding zero-mode of $W_d(x)$ we consider

$$\det W_d(x) = -(d-1)|x_d|^2 \left(\sum_{i=1}^{d-1} |x_i|^2 \right) \cdot z^{d-2} + (d-1)|x_d|^2 z^{d-1} = 0 \ .$$

Observing that the last row of $W_d(x)$ is a combination of the previous ones, we find the vector of the kernel solving the equation

$$[zI|\vec{a}] \left[\begin{array}{c} \vec{v} \\ w \end{array} \right] = z\vec{v} + \vec{a}w = 0 \tag{9}$$

which implies (up to a scalar), that $\vec{v} = \vec{a}$ and $w = -z$. Denoting the solution as $y(x)$, one gets the family $q(x) = x \otimes y(x)$ of product vectors such that $\langle x \otimes y(x)|W|x \otimes y(x)\rangle = 0$. A vector from the family has the following coordinates:

$$\begin{array}{cccccc} x_1x_1x_d^* & \ldots & x_1x_{d-2}x_d^* & x_1x_{d-1}^*x_d & x_1 \sum_{i=1}^{d-1} x_ix_i^* \\ x_2x_1x_d^* & \ldots & x_2x_{d-2}x_d^* & x_2x_{d-1}^*x_d & x_2 \sum_{i=1}^{d-1} x_ix_i^* \\ \vdots & \vdots & & \vdots & \\ x_dx_1x_d^* & \ldots & x_dx_{d-2}x_d^* & x_dx_{d-1}^*x_d & x_d \sum_{i=1}^{d-1} x_ix_i^* \ . \end{array}$$

It remains to show that vectors $q(x) = x \otimes y(x)$ span $\mathbb{C}^d \otimes \mathbb{C}^d$. Suppose that there exists a vector $\alpha = \sum_{i,j=1^d} \alpha_{i,j}|e_i\rangle \otimes |e_j\rangle$ orthogonal to $q(x)$ for all x, that is,

$$\sum_{i=1}^{d} \left(\sum_{j=1}^{d-2} \alpha_{i,j}^* x_i x_j x_d^* + \alpha_{i,d-1}^* x_i x_{d-1}^* x_d + \alpha_{i,d}^* x_i \left(\sum_{i=1}^{d-1} x_i x_i^* \right) \right) = 0 \ .$$

We stress that in the linear space of polynomials of $2d$ variables x_i and x_i^* are linearly independent. The monomial $x_ix_1x_1^*$ appears in the sum only once multiplied by the coefficient $\alpha_{i,d}$. Hence because different monomials are linearly independent in the space of polynomials one concludes that $\alpha_{i,d} = 0$. Next observe, that the monomial $x_ix_{d-1}^*x_d$ appears only once multiplied by the coefficient $\alpha_{i,d-1}$. Thus one concludes that $\alpha_{i,d-1} = 0$. Finally, we have to prove, that the sum $\sum_{i=1}^d \sum_{j=1}^{d-2} \alpha_{i,j}^* x_i x_j x_d^*$ is zero iff all coefficients are zero. Indeed, all the coefficients multiply the different monomials. There are no non-zero vectors orthogonal to the subspace spanned by the vectors $q(x)$, so these vectors span the whole Hilbert space of the system, what implies optimality of the witness. $\square$

In conclusion we have shown how to generalize a positive map in $M_3(\mathbb{C})$ considered in [5] to a positive map in $M_d(\mathbb{C})$. We have proved that this map is optimal and indecomposable. As a byproduct we provide a class of PPT entangled states in $d \otimes d$. It would be interesting check whether this generalized map is extremal or even exposed.

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